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$\boxed{\text{Singular stochastic nonlinear wave equations}}$

Stochastic nonlinear wave equation (SNLW)

with an additive space-time white noise.

$$(\text{SNLW}) \quad \begin{cases} (\partial_t^2 - \Delta) u + u^k = \xi \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^d) \end{cases}$$

on $\mathbb{R}_+ \times \mathbb{T}^d$, $\quad \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$

$$\mathcal{H}^s(\mathbb{T}^d) = H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d)$$

- ξ = space-time white noise

  = Gaussian process indexed by (t, x) s.t.

$$\mathbb{E}\big[\xi(t_1, x_1)\xi(t_2, x_2)\big] = \delta(t_1 - t_2)\,\delta(x_1 - x_2)$$

- Ito formulation: With $v = \partial_t u$,

$$d\begin{pmatrix} u \\ v \end{pmatrix} + \left\{ \begin{pmatrix} 0 & -1 \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ u^k \end{pmatrix} \right\} dt = \begin{pmatrix} 0 \\ dW \end{pmatrix}.$$

Here, $W = L^2$- cylindrical Wiener process

$$W(t) = \sum_{n \in \mathbb{Z}^d} \beta_n(t)\, e_n$$

- $e_n(x) = \frac{1}{(2\pi)^{d/2}} e^{in \cdot x}$
- $\{\beta_n\}_{n \in \mathbb{Z}^d}$ = family of mutually independent complex-valued Brownian motions

conditioned such that $\beta_{-n} = \overline{\beta_n}$, $n \in \mathbb{Z}^d$

"$\Lambda = \mathbb{Z}^d/2$"

convention : $\mathrm{Var}(\beta_n(t)) = t$.

i.e. $\beta_n = \underbrace{\mathrm{Re}\,\beta_n + i\,\mathrm{Im}\,\beta_n}_{}$ indep real B.M. $/\sqrt{2}$

$$\Lambda = \bigcup_{k=0}^{d-1} \left(\mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1}\right)$$

Then, $\{\beta_n\}_{n \in \Lambda}$ is independent.

Duhamel formulation (= mild formulation)

We say that u is a soln to SNLW if u satisfies

$$u(t) = \partial_t S(t) u_0 + S(t) u_1 - \int_0^t S(t-t') u^k(t')\,dt' + \boxed{\underbrace{\int_0^t S(t-t')\,dW(t')}_{\text{stochastic convolution.}}}$$

where $S(t) = \dfrac{\sin(t|\nabla|)}{|\nabla|}$, $\quad |\nabla| = \sqrt{-\Delta}$

$$\widehat{S(t) f}(n) = \begin{cases} \dfrac{\sin(t|n|)}{|n|} \hat{f}(n), & n \neq 0 \\ t\, \hat{f}(n), & n = 0. \end{cases}$$

In the following, we actually study

$$\begin{cases} (\partial_t^2 + 1 - \Delta) u + u^k = \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

corresponding to the Klein-Gordon eqn but we simply refer to this as SNLW.

Duhamel formula:

$$u(t) = \partial_t S(t) u_0 + S(t) u_1 - \int_0^t S(t-t') u^k(t') dt'$$
$$+ \int_0^t S(t-t') dW(t')$$

Now, $S(t) = \dfrac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}$, $\quad \langle\nabla\rangle = \sqrt{1-\Delta}$

$\langle\cdot\rangle = \sqrt{1-|\cdot|^2}$

= Japanese bracket.

· The only difference appears at the zeroth freq. but it does not affect well-posedness theory

• Stochastic $\boxed{\text{damped}}$ NLW (SdNLW)

$$\begin{cases} (\partial_t^2 + \partial_t + 1 - \Delta) u + u^k = \sqrt{2}\,\xi \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1) \end{cases}$$

Ito formulation:

$$d\begin{pmatrix} u \\ v \end{pmatrix} + \left\{ \begin{pmatrix} 0 & -1 \\ 1-\Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ u^k \end{pmatrix} \right\} dt$$
$$= \begin{pmatrix} 0 \\ -v \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2}\, dW \end{pmatrix}$$

Duhamel formulation:

$$u(t) = \partial_t \mathcal{D}(t) u_0 + \mathcal{D}(t)(u_0 + u_1)$$
$$- \int_0^t \mathcal{D}(t-t') u^k(t') dt'$$
$$+ \sqrt{2} \int_0^t \mathcal{D}(t-t') dW(t'),$$

where $\mathcal{D}(t) = e^{-\frac{t}{2}} \dfrac{\sin\left(t\sqrt{\frac{3}{4}-\Delta}\right)}{\sqrt{\frac{3}{4}-\Delta}}.$

$\Leftarrow$ In terms of local well-posedness, there is <u>no</u> difference from SNLW (either wave or KG.)

Why care?

① $\xi = dW$ has regularity

$-\frac{d}{2} - \varepsilon$ in x

$-\frac{1}{2} - \varepsilon$ in t, very rough.

expect $\Rightarrow$ stoch. convolution $\Psi = \int_0^t \dfrac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} dW(t')$

has spatial regularity

$1 - \frac{d}{2} - \varepsilon$ (< 0 for $d \geq 2$).

$\Rightarrow$ analytically challenging.

③ SdNLW formally preserves the Gibbs measure ρ. $(k \in 2\mathbb{N}+1)$

$$d\rho = Z^{-1} e^{-H(u,v)} du \otimes dv$$

$$= Z^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}^d} u^{k+1} dx} e^{-\frac{1}{2}\|u\|_{H^1}^2} du$$

$$\otimes \; e^{-\frac{1}{2}\|v\|_{L^2}^2} dv$$

$$= \underbrace{Z^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}^d} u^{k+1} dx} d\mu_1(u)}_{\Phi_d^{k+1}\text{- measure}} \otimes d\mu_0(v),$$

where

- $H(u,v)$ = Hamiltonian (= energy) for NLW/NLKG

$$= \frac{1}{2}\int_{\mathbb{T}^d} |\langle\nabla\rangle u|^2 dx + \frac{1}{2}\int_{\mathbb{T}^d} v^2 dx + \frac{1}{k+1}\int_{\mathbb{T}^d} u^{k+1} dx.$$

- μ_s = Gaussian meas with CM space $H^s(\mathbb{T}^d)$

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du.$$

$$\Longleftrightarrow \quad u = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x}$$

$\{g_n\}$, indep standard $\mathbb{C}$-valued Gaussian r.v.'s s.t. $g_{-n} = \overline{g_n}$.

SdNLW

$$\underbrace{d\begin{pmatrix}u\\v\end{pmatrix} + \left\{\begin{pmatrix}0 & -1\\ 1-\Delta & 0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}0\\u^k\end{pmatrix}\right\}dt}_{\text{NLW in a vectorial form}}$$

$$= \underbrace{\begin{pmatrix}0\\-v\end{pmatrix}dt + \begin{pmatrix}0\\ \sqrt{2}\,dW\end{pmatrix}}_{\text{Ornstein-Uhlenbeck process (for } v)}.$$

- NLW preserves the Gibbs measure ρ.

  1-d : Friedlander '85.
  2-d : Oh - Thomann '17.

- OU preserves the white noise μ_0 for v and hence preserves the Gibbs measure $\rho(du, dv)$.

- $\mathcal{L}$ = generator for SdNLW

  $= \mathcal{L}_1 + \mathcal{L}_2$

  $\mathcal{L}_1$ = generator for NLW
  $\mathcal{L}_2$ = generator for OU.

ρ invariant $\iff \mathcal{L}^* \rho = 0$

i.e. $\int \mathcal{L} F(u,v)\, d\rho(u,v) = 0$

$\Leftarrow \mathcal{L}_1^* \rho = 0$ AND $\mathcal{L}_2^* \rho = 0$.

b/c $\mathcal{L}^* = \mathcal{L}_1^* + \mathcal{L}_2^*$

This can be made rigorous by considering the "finite-dimensional" approximation:

(SdNLW$_N$) $$(\partial_t^2 + \partial_t + 1 - \Delta) u^N + P_{\le N}\left((P_{\le N} u^N)^k \right) = \sqrt{2}\, \xi$$

$\Leftarrow$ (i) finite dim'l SdNLW on $\{|n| \le N\}$ decoupled
(ii) linear stochastic damped wave dynamics on high freq

i.e. this preserves

$$d\rho_N = Z_N^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}^d} (P_{\le N} u)^{k+1} dx}\, d\mu_1(u) \otimes d\mu_0(v)$$

$$= Z_N^{-1} \underbrace{\left(e^{-\frac{1}{k+1}\int (P_{\le N} u)^{k+1} dx}\, d\mu_{1,N}(u) \otimes d\mu_{0,N}(v) \right)}_{(i)} \otimes \underbrace{\left(d\mu_{1,N}^{\perp}(u) \otimes d\mu_{0,N}^{\perp}(v) \right)}_{(ii)}$$

Main Goal:

Prove local well-posedness of SdNLW along with a good approximation property by SdNLW_N.

$\Rightarrow$ Then, we can apply Bourgain's invariant measure argument to obtain

- almost sure global well-posedness of SdNLW
- invariance of the Gibbs measure ρ.

$$\begin{tikzcd} \rho_N \arrow[r] \arrow[d, dash] & \rho \arrow[d, dash, dotted] & \leftarrow \text{inv.} \\ u^N \arrow[r] & u & \end{tikzcd}$$

See Bourgain '94, '96, Burq-Tzvetkov '08.

Idea: Use formal invariance of ρ as a replacement of a conservation law.

(· parabolic Φ^4_3-model: Hairer-Matetski '18.

$\Rightarrow$ shows SdNLW is a stochastic quatization equation (SQE) for

Φ^{k+1}_d - measure for u $\otimes$ white noise μ_0 for v.

- SdNLW is called the canonical stochastic quantization equation. (Ryang, Saito, Shigemoto '85.)

$\Leftarrow$ hyperbolic Φ^{k+1}_d - model.

Parabolic counterpart:

Parabolic Φ^{k+1}_d - model / stochastic quantization eqn

$$(\partial_t - \Delta) u + u^{k+1} = \xi \quad \text{on } \mathbb{T}^d$$

- $d = 2$: Da Prato - Debussche '03.
- $d = 3$: ($k = 3$)

Hairer '15: regularity structure

Gubinelli - Imkeller - Perkowski '15: paracontrolled distributions (Catellier - Chouk '18)

Kupiainen '16: renormalization group method

. In the following, we focus on local well-posedness

$\Rightarrow$ We only consider the (second) SNLW.

Chap 1: SNLW in 2-d.

Chap 2: quadratic SNLW in 3-d.

Chap 3: cubic SNLW of Hartree-type in 3-d
(with Mamoru Okamoto)

$\Leftarrow$ I did not get to cover this part.

Chapter 0: Preliminaries

(SNLW)

$$u(t) = \partial_t S(t) u_0 + S(t) u_1 - \int_0^t S(t-t') u^k(t') dt' + \Psi$$

Ψ = stochastic convolution

$$= \int_0^t S(t-t') dW(t')$$

$$= \sum_{n \in \mathbb{Z}^d} e_n \overbrace{\int_0^t \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle} d\beta_n(t')}^{\text{Wiener integral}}$$

• Wiener integral ($\mathbb{R}$-valued case)

Given deterministic $f \in L^2([a,b])$,

define $I(f) = \int_a^b f\, dB$ by the left endpt Riemann sum.

$\Rightarrow$ $I(f)$ is a mean 0 Gaussian r.v.

with $\mathrm{Var}(I(f)) = \|f\|^2_{L^2([a,b])}$.

i.e. $I : L^2([a,b]) \to L^2(\Omega)$ is an isometry (onto its image)

• The following lemma will be useful in studying regularities of various random distributions

• $\mathcal{H}_k$ = homogeneous Wiener chaoses of order k.

$\mathcal{H}_{\le k} = \bigoplus_{j=1}^{k} \mathcal{H}_j$.

(Think of the $L^2(\Omega)$-completion of polynomials in B_n's of degree $\le k$.

Lemma 1 (Wiener chaos estimate)

Let $k \in \mathbb{N}$. Then, we have

$$\|X\|_{L^p(\Omega)} \leq (p-1)^{k/2} \|X\|_{L^2(\Omega)}$$

for any $p \geq 2$ and $X \in \mathcal{H}_{\leq k}$.

($\Leftarrow$ Nelson's hypercontractivity of the OU semigroup.

Lemma 2: Let $\{X_N\}_{N \in \mathbb{N}}$ and X be spatially homogenous stochastic processes: $\mathbb{R}_+ \to \mathcal{D}'(\mathbb{T}^d)$

i.e. for any $x_0 \in \mathbb{T}^d$,

$$\{X(\cdot, t)\}_{t \in \mathbb{R}_+} \text{ and } \{X(x_0 + \cdot, t)\}_{t \in \mathbb{R}_+}$$

have the same law.

- Suppose that $\exists\, k \in \mathbb{N}$ s.t. $X_N(t)$ and $X(t)$ belong to $\mathcal{H}_{\leq k}$ for each $t \in \mathbb{R}_+$.

Then,

(i) $t \in \mathbb{R}_+$. If $\exists\ s_0 \in \mathbb{R}$ such that

(*) $$\mathbb{E}\left[|\hat{X}(t,n)|^2\right] \lesssim \langle n \rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$, then

$$X(t) \in W^{s,\infty}(\mathbb{T}^d), \quad s < s_0, \quad a.s.$$

Bessel potential space

$$\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p}$$

$$= \|\mathcal{F}^{-1}(\langle n \rangle^s \hat{f}(n))\|_{L^p}.$$

Moreover, if $\exists\ \theta > 0$ s.t.

$$\mathbb{E}\left[|\hat{X}_N(t,n) - \hat{X}(t,n)|^2\right] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$ and $N \geq 1$, then

$$X_N \to X \text{ in } W^{s,\infty}(\mathbb{T}^d), \quad s < s_0, \quad a.s.$$

(ii) Given $h \in \mathbb{R}$, define the difference operator δ_h by $\delta_h X(t) = X(t+h) - X(t)$.

Let $T > 0$ and suppose (i) holds on $[0,T]$.

If $\exists\ \sigma \in (0,1)$ s.t.

$$\mathbb{E}\left[|\delta_h \widehat{X}(t,n)|^2 \right] \lesssim \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma$$

for any $n \in \mathbb{Z}^d$, $t \in [0,T]$, $|h| \leq 1$ with $0 \leq t+h \leq T$,

then, $\quad X \in C([0,T]; W^{s,\infty}(\mathbb{T}^d))$

for $s < s_0 - \sigma/2$, a.s.

Futhermore, if $\exists\ \theta > 0$ s.t.

$$\mathbb{E}\left[|\delta_h \widehat{X}_N(t,n) - \delta_h \widehat{X}(t,n)|^2 \right] \lesssim N^{-\theta} \langle n \rangle^{-d-2s_0+\sigma} |h|^\sigma,$$

for any $n \in \mathbb{Z}^d$, $t \in [0,T]$, $|h| \leq 1$, and $N \geq 1$,

then X_N converges to X in $C([0,T]; W^{s,\infty}(\mathbb{T}^d))$

for $s < s_0$, a.s.

See Mourrat-Weber-Xu, Oh-Okamoto-Tzvetkov

<u>Idea of the proof:</u>

translation invariance

$$\Rightarrow \quad \mathbb{E}\left[\widehat{X}(t,n) \widehat{X}(t,m) \right] = 0$$

for $n + m \neq 0$.

(real-valued setting.)

$$\cdot\ \mathbb{E}\left[\hat{X}(t,n)\hat{X}(t,m)\right]$$

$$= \iint \underbrace{\mathbb{E}\left[X(t,x)X(t,y)\right]}_{=F(t,x-y)} \underbrace{e_{-n}(x)\, e_{-m}(y)}_{=e_{-(n+m)}(x)\, e_m(x-y)}\, dy\, dx$$

$$= \hat{F}(t,m) \int_{\mathbb{T}^d} e^{-i(n+m)\cdot x}\, dx$$

$$= 0 \quad \text{for } n+m \neq 0.$$

Suppose ⊛ holds. Then, for $p \gg 1$,

Suppress t

$$\Big\| \|X\|_{W^{s,\infty}} \Big\|_{L^p(\Omega)} \overset{\text{Sobolev}}{\lesssim} \Big\| \|X\|_{W^{s+\varepsilon,p}} \Big\|_{L^p(\Omega)}$$

$$= \Big\| \| \langle\nabla\rangle^{s+\varepsilon} X(x) \|_{L^p(\Omega)} \Big\|_{L^p_x}$$

$$\overset{\text{Lem 1}}{\lesssim} p^{k/2} \Big\| \underbrace{\Big\| \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{s+\varepsilon} \hat{X}(n)\, e_n(x) \Big\|_{L^2(\Omega)}}_{=\left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2(s+\varepsilon)} |\hat{X}(n)|^2\right)^{1/2}} \Big\|_{L^p_x}$$

$$\leq C_p \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2(s+\varepsilon)-d-2s_0} \right)^{1/2} < \infty$$

iff $s+\varepsilon-s_0 < 0$ i.e. $s < s_0$. $\boxtimes$

Lemma 3: $\Psi \in C(\mathbb{R}_+ ; W^{1-\frac{d}{2}-,\infty}(\mathbb{T}^d))$, a.s.

Moreover,

$$\Psi_N = P_{\le N}\Psi \to \Psi$$

in $C(\mathbb{R}_+ ; W^{1-\frac{d}{2}-,\infty}(\mathbb{T}^d))$, a.s.

$\uparrow$ compact-open topology

Pf: We only verify ⊛ in Lemma 2.

$$\mathbb{E}\left[|\widehat{\Psi}(t,n)|^2\right] = \int_0^t \left(\frac{\sin((t-t')\langle n\rangle)}{\langle n\rangle}\right)^2 dt'$$

$$\lesssim_t \langle n\rangle^{-2} = \langle n\rangle^{-d-2\underbrace{(1-\frac{d}{2})}_{=s_0}}.$$

- For temporal regularity (ii), use mean value thm in time $\boxtimes$

- When $d=1$,

$$\Psi \in C(\mathbb{R}_+ ; W^{\frac{1}{2}-,\infty}(\mathbb{T}^d)), \text{ a.s.}$$

$\Rightarrow$ LWP of SNLW is trivial.

- For $d \ge 2$, $\Psi(t)$ is NOT a function, only a distribution.

$\Rightarrow$ $\exists$ issue in making sense of $(\Psi(t))^k$ (and hence u^k.)

$\Rightarrow$ Need to renormalize the nonlinearity

Chap 1: SNLW in 2-d

$$u(t) = \partial_t S(t) u_0 + S(t) u_1 - \int_0^t S(t-t') u^k(t') dt' + \Psi.$$

Picard 2nd iterate: $\int_0^t S(t-t') \Psi^k(t') dt'$

does NOT make sense.

Triviality: u_ε = soln to SNLW with regularized noise $\xi_\varepsilon = \eta_\varepsilon * \xi$.

Without a proper renormalization,

$u_\varepsilon \to$ lin soln (or 0)

Albeverio-Haba-Russo '96

Oh-Okamoto-Robert '19.

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i.e. renormalization is necessary in order to have a non-trivial limit.

Truncated SNLW:

$$(\partial_t^2 + 1 - \Delta) u_N + u_N^k = P_{\leq N} \xi.$$

$\Rightarrow$ Truncated convolution $\Psi_N = P_{\leq N} \Psi$.

smooth in x.

Fix $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^2$.

$\Psi_N(t,x)$ = mean 0 Gaussian r.v. with

variance $\sigma_N(t) = \mathbb{E}[\Psi_N^2(t,x)]$

$$= \sum_{|n| \le N} \int_0^t \frac{(\sin((t-t')\langle n \rangle))^2}{\langle n \rangle^2} dt'$$

$$\sim t \log N \to \infty \quad \text{as } N \to \infty.$$

Da Prato - Debussche trick ('02 - '03)
McKean '95
Bourgain '96

Duhamel $\Rightarrow u_N = v_N + \Psi_N$

postulate v_N is smoother.

Binomial:

$$u_N^k = \sum_{j=0}^{k} \binom{k}{j} \Psi_N^j v_N^{k-j}$$

does NOT have a limit (as $N \to \infty$)

Wick renormalization: Replace Ψ_N^j by

$$:\Psi_N^j(t,x): = H_j(\Psi_N(t,x); \sigma_N(t))$$

ptwise operation. time-dependent

$H_j(x;\sigma)$ = Hermite poly of deg j.

defined by the generating function:

$$G(t,x;\sigma) = e^{tx - \frac{1}{2}\sigma t^2} = \sum_{j=0}^{\infty} \frac{t^j}{j!} H_j(x;\sigma)$$

Prop 4: $T > 0$.

$\{:\Psi_N^j:\}_{N \in \mathbb{N}}$ is Cauchy in $L^p(\Omega; C_T W_x^{-\varepsilon,\infty})$
$p < \infty$

also almost surely in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$

Pf: We only verify $\circledast$ in Lemma 2.
(unif in $N \geq 1$.)

Recall: For mean 0 Gaussian r.v.'s f and g with variances σ_f and σ_g, we have

$$\mathbb{E}[H_j(f;\sigma_f) H_k(g;\sigma_g)]$$
$$= \delta_{jk} \cdot j! \{\mathbb{E}[fg]\}^j.$$

$\Leftarrow$ follows from computing

$$\int_\Omega G(t,f;\sigma_f)\, G(s,g;\sigma_g)\, dP$$

- directly
- in terms of Hermite poly

$\}$ compare coeff.

$$\mathbb{E}\Big[\big|\widehat{:\Psi_N^j(t,n):}\big|^2\Big]$$

$$= \iint \underbrace{\mathbb{E}\big[:\Psi_N^j(t,x): \; :\Psi_N^j(t,y):\big]} e_n(y-x)\,dx\,dy$$

$$= j!\,\Big\{\underbrace{\mathbb{E}\big[\Psi_N(t,x)\Psi_N(t,y)\big]}\Big\}^j$$

$$= \sum_{\substack{m\in\mathbb{Z}^2\\ |m|\le N}} \int_0^t \frac{\big(\sin((t-t')\langle m\rangle)\big)^2}{\langle m\rangle^2}\,dt'\, e_m(x-y)$$

$$\lesssim \sum_{n_1+\cdots+n_j=n} \frac{1}{\langle m_1\rangle^2\cdots\langle m_j\rangle^2} \lesssim \frac{1}{\langle m\rangle^{2-\varepsilon}}.$$

As before, the time difference in (ii) can be established by MVT $\boxtimes$

Define

$$:u_N^k: = \sum_{j=0}^{k}\binom{k}{j} :\Psi_N^j:\; v_N^{k-j}$$

$$\downarrow$$

$$:u^k: = \sum_{j=0}^{k}\binom{k}{j} :\Psi^j:\; v^{k-j}$$

smoother.

for $u = \Psi + v$

$\Rightarrow$ Renormalized SNLW (rSNLW)

$$\begin{cases} (\partial_t^2 + 1 - \Delta) u + :u^k: = \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}.$$

Really means

$$(\ast\ast) \quad \begin{cases} (\partial_t^2 + 1 - \Delta) v + \sum_{j=0}^{k} \binom{k}{j} :\Psi^j: v^{k-j} = 0 \\ (v, \partial_t v)|_{t=0} = (u_0, u_1). \end{cases}$$

Theorem 5 : (Gubinelli - Koch - Oh '18)

Let $k \geq 2$. Then, rSNLW is locally well-posed in $\mathcal{H}^s(\mathbb{T})$ for

(i) $k \geq 4$: $s \geq s_{crit} = \max\Big(\underbrace{1 - \frac{2}{k-1}}_{\text{scaling}}, \underbrace{\frac{3}{4} - \frac{1}{k-1}}_{\text{conformal}}, 0\Big)$

(ii) $k = 2, 3$: $s > s_{crit}$.

Idea : Solve the fixed pt problem $(\ast\ast)$ for v by

1. Strichartz estimates (on lin solns)
2. Prop 4 on $:\Psi^j:$
3. product estimates.

(i) fractional Leibniz rule.

(ii) $\frac{1}{p} + \frac{1}{q} \le \frac{1}{r} + \frac{s}{d}$

$$\Rightarrow \| \langle \nabla \rangle^{-\varepsilon} (fg) \|_{L^r} \lesssim \| \langle \nabla \rangle^{-\varepsilon} f \|_{L^p} \| \langle \nabla \rangle^{\varepsilon} g \|_{L^q}$$

A typical term

$$\| \langle \nabla \rangle^{-\varepsilon} (:\Psi^j: v^{k-j}) \|$$

$$\underset{(ii)}{\lesssim} \| \langle \nabla \rangle^{-\varepsilon} :\Psi^j: \| \underbrace{\| \langle \nabla \rangle^{\varepsilon} (v^{k-j}) \|}_{\underset{(i)}{\lesssim} \| \langle \nabla \rangle^{\varepsilon} v \| \, \| v \|^{k-j-1}}$$

$$\begin{aligned} u &= \Psi + v \\ &\in \Psi + C([0,T]; H^\sigma(\mathbb{T}^2)), \quad \sigma = s \wedge (1-\varepsilon) \\ &\subset C([0,T]; H^{-\varepsilon}(\mathbb{T}^2)). \quad \forall \varepsilon > 0 \end{aligned}$$

Back to page 9 : LWP of renormalized SdNLW

$\Rightarrow$ a.s. GWP of renormalized SdNLW
(= hyperbolic Φ_2^{k+1}-model)

and invariance of the Gibbs measure

(by applying Bourgain's inv. meas. argument.

- unique ergodicity (cubic) : L. Tolomeo '19.

Q: What about (deterministic) global well-posedness?

- In the heat case, one simly estimates $\partial_t \|v\|^p_{L^p}$.
- In the dispersive setting, we need to rely on the energy but the situation is much more intricate.

We only consider the defocusing cubic case.

$$(\partial_t^2 + 1 - \Delta) v + v^3 + \underbrace{3 v^2 \Psi + 3 v :\Psi^2: + :\Psi^3:}_{\text{rough perturbation}} = 0$$

Two difficulties:

① $v(t) \notin H^1(\mathbb{T}^2)$.

$\Rightarrow$ can not use the energy

$$H(v, \partial_t v) = \frac{1}{2}\int |\langle\nabla\rangle v|^2 dx + \frac{1}{2}\int (\partial_t v)^2 + \frac{1}{4}\int v^4 dx.$$

- We need to smooth v.

$\Rightarrow$ I-method in the stochastic setting.

② Even if v were in H^1, v does not satisfy (deterministic) NLW, i.e. $H(v)$ is NOT conserved.

- If a noise is a bit smoother

($\text{think of } \langle\nabla\rangle^{-2}\Psi \in C_t L^2_x$) s.t. $v \in C_t H^1_x$

then

$$\partial_t H(v) = \int_{\mathbb{T}^2} \partial_t v \, \big(\underbrace{(\partial_t^2 + 1 - \Delta) v}_{= -(v+\Psi)^3} + v^3 \big) \, dx$$

← No need for renormalization

$$\sim v^2 \Psi + v \Psi^2 + \Psi^3$$

v^3-term is gone!!

$$\overset{C\text{-}S}{\lesssim} (H(v))^{1/2} \Big(\|\Psi\|^2_{C_T L^\infty_x} \int v^4 dx + \|\Psi\|^6_{C_T L^6_x} \Big)^{1/2}$$

$$\le C(T, \Psi) H(v)$$

$\Longrightarrow$ Gronwall. (Burq-Tzvetkov '14 in the random data well-posedness theory on 3-d cubic NLW

- I-method (= method of almost conservation law

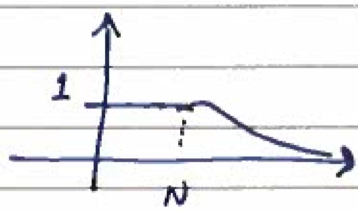
  - CKSTT '02

    (after Bourgain's high-low method '98

Let $N \in \mathbb{N}$. Define

$0 < s < 1$

$$m_N(n) = \begin{cases} 1, & |n| \le N \\ \dfrac{N^{1-s}}{|n|^{1-s}}, & |n| > 2N \end{cases}$$



Set

$I = I_N$ = Fourier multiplier op with m_N.

low freq : identity

high freq : fractional integration

def & LP theory

- $$\|If\|_{W^{a+\sigma,p}} \lesssim N^{\sigma} \|f\|_{W^{a,p}}$$

  $\forall \ 0 \le \sigma \le 1-s$

  $\forall \ 1 < p < \infty$

- $$\|f\|_{H^s} \lesssim \|If\|_{H^1} \lesssim N^{1-s} \|f\|_{H^s}.$$

$\Rightarrow$ Iv does not satisfy NLW in 2 ways

① b/c of I

② b/c of the terms with Ψ.

$H(Iv)(t) - H(Iv)(0)$

$$= \int_0^t \int (\partial_t Iv)\Big(\underbrace{(\partial_t^2 + 1 - \Delta) Iv}_{\substack{= -I(:(v+\Psi)^3:) \\ = -I(v^3 + 3v^2\Psi + 3v:\Psi^2: + :\Psi^3:)}} + (Iv)^3\Big)$$

$$= \int_0^t \int_x (\partial_t Iv)\big(-Iv^3 + (Iv)^3\big)$$

I-method part (commutator part)

$$\left.\begin{aligned} &- 3\int_0^t \int_x (\partial_t Iv)\, I(v^2\Psi) \\ &- 3\int_0^t \int_x (\partial_t Iv)\, I(v:\Psi^2:) \\ &- \int_0^t \int_x (\partial_t Iv)\, I(:\Psi^3:) \end{aligned}\right\}\ \text{Gronwall part}$$

$$=: A_1 + A_2 + A_3 + A_4$$

Theorem 6: (GKO-Tolomeo '18)

The defocusing cubic rSNLW is globally well-posed in $\mathcal{H}^s(\mathbb{T}^2)$, $s > 4/5$.

Sketch of the proof:

① Commutator estimates

Lemma 7: $k \le 3$

$$\| (If)^k - I(f^k) \|_{L^2} \lesssim N^{-\frac{1-k(1-s)}{2}} \| If \|_{H^1}^k$$

one of the freq. must be high.

Lemma 8: Given $\gamma > 0$, $0 < \sigma < 1$,

$\exists\ \eta(\gamma) > 0$ and $p(\gamma) \gg 1$ s.t.

$$\| (If)(Ig) - I(fg) \|_{L^2} \lesssim N^{\gamma - \frac{1-\sigma}{2}} \| f \|_{H^{1-\sigma}} \| g \|_{W^{-\eta, p}}.$$

Lemma 9: $k = 0, 1, 2$. $\forall \gamma$, $\exists\ \eta, p$ s.t.

$$\| I(v^k : \Psi^{3-k} :) - (Iv)^k I(: \Psi^{3-k} :) \|_{L^2}$$
$$\lesssim N^{-\frac{1-k(1-s)}{2} + \gamma} \| Iv \|_{H^1}^k \| : \Psi^{3-k} : \|_{W^{-\eta, p}}$$

($\Leftarrow$ Lemmas 7 & 8

Note: We may lower the regularity by using spacetime estimates.

② On the stochastic term:

Lemma 10: $p < \infty$

$$\left(\mathbb{E} \| I \Psi \|_{L^p_{T,x}}^p \right)^{1/p} \lesssim p^{1/2} T^{1/2 + 1/p} (\log N)^{1/2}.$$

Pf: Separately estimate $\underbrace{I P_{\le N} \Psi}_{= P_{\le N} \Psi}$ and $I P_{> N} \Psi$.

$\Rightarrow$ Fubini & Chebyshev,

$$P\left(\| I \Psi \|_{L^p_{T,x}} > \lambda \right) \lesssim \frac{p^{p/2} T^{\frac{p}{2}+1} (\log N)^{p/2}}{\lambda^p}.$$

$\boxtimes$

For Gronwall part, we have

Lemma 11 : (i) $k=0,1$, $\forall\, 0<\theta\leq 1-s$

$$\left|\int_{\mathbb{T}^2}(\partial_t Iv)(Iv)^k I(:\Psi^{3-k}:)\right|$$

L^2_x L^4_x

$$\lesssim N^\theta\left(1+H(Iv)^{\frac{3}{4}}\right)\|:\Psi^{3-k}:\|_{L^\infty_T W^{-\theta,4}_x}$$

BAD

mapping property of I.

(ii) ($k=2$) $\exists\, c>0$ s.t. $\forall\, 0<\gamma<1/8$,

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{T}^2}(\partial_t Iv)(Iv)^2(I\Psi)\,dx\,dt\right|$$

$$\lesssim\left(1+(t_2-t_1)+\int_{t_1}^{t_2}H(v)^{1+c\gamma}\right)\|I\Psi\|_{L^{\gamma^{-1}}_{(t_1,t_2),x}}.$$

($\Leftarrow$ interpolation & Sobolev .

Putting together, we have

$$H(Iv)(t) - H(Iv)(t_0)$$

C-S

$$= \int_{t_0}^{t} \int_x (\partial_t Iv) \underbrace{\left(-Iv^3 + (Iv)^3 \right)}_{\text{Lemma 7}}$$

$$- 3 \int_{t_0}^{t} \int_x (\partial_t Iv) \Big(\underbrace{I(v^2 \Psi) - (Iv)^2 I\Psi}_{\text{Lemma 9}} \Big)$$

commutator terms

$$- 3 \int_{t_0}^{t} \int_x (\partial_t Iv)(Iv)^2 I\Psi$$

Lemma 11(ii)

$$- 3 \int_{t_0}^{t} \int_x (\partial_t Iv) \underbrace{\Big(I(v : \Psi^2 :) - (Iv) I(: \Psi^2 :) \Big)}_{\text{Lemma 9}}$$

Gronwall part

$$\left. \begin{aligned} & - 3 \int_{t_0}^{t} \int_x (\partial_t Iv)(Iv) I(: \Psi^2 :) \\ & - \int_{t_0}^{t} \int_x (\partial_t Iv) I(: \Psi^3 :) \end{aligned} \right\} \text{Lemma 11(i)}$$

Given $T>0$ and $\theta>0$, define

$$A(N) = \| I\Psi \|_{L^{\log N}_{T,x}} \Big/ \log N$$

commutator part

$$\Omega_{M,\gamma,\theta} = \Big\{ \max_{k=1,2} \| :\Psi^{3-k}: \|_{L^\infty_T W_x^{-\gamma(\sigma),p(\sigma)}}$$

$$+ \max_{k=0,1} \| :\Psi^{3-k}: \|_{L^\infty_T W_x^{-\theta,4}} \leq M \Big\}$$

Gronwall part.

Prop 12: $\exists\ \gamma = \gamma(s) > 0,\quad \alpha = \alpha(s) > 0$
$\theta = \theta(s) > 0,$

$\beta = \beta(s) > 0$ with $\beta < \alpha \leq 1 - 3(1-s)$

s.t. if $\omega \in \Omega_{M,\gamma,\theta}$, $H(Iv)(t_0) \leq N^\beta$
$|t_0| \leq T,$

then $\exists\ \tau = \tau(M,T,s)$

$\delta = \delta(s) > 0,\quad N_0 = N_0(T,s)$

s.t.

$$H(Iv)(t) \leq N^\alpha$$

provided that $|t - t_0| \leq \tau \big/ A(N)$

$|t| < T$

$A(N) < N^\delta,\quad N \geq N_0$

<u>Pf:</u> $H(t) - H(t_0) \lesssim \int_{t_0}^{t} N^{-(1-3(1-s))} H^2$ comm part

$$+ \sum_{k=1}^{2} \int_{t_0}^{t} N^{\frac{-1-k(1-s)}{2}+\gamma} H^{\frac{k+1}{2}} \| :\Psi^{3-k}: \|_{L^\infty_T W_x^{-\gamma(r),p(r)}}$$

$$+ \sum_{k=0}^{1} \int_{t_0}^{t} N^{\theta} (1 + H^{3/4}) \| :\Psi^{3-k}: \|_{L^\infty_T W_x^{-\theta,4}}$$

$$+ \left(1 + (t-t_0) + \int_{t_0}^{t} H^{1+c\eta} \right) \| I\Psi \|_{L^{\eta^{-1}}_{T,x}}$$

Choose $\eta = (\log N)^{-1}$

$F(t) \geq H(t)$ and $N^\beta \leq F \leq N^\alpha$.

(i) $N^{-(1-3(1-s))} H^2$

$\leq N^{-\alpha} F \cdot F \leq F$, $\quad \alpha \leq 1-3(1-s)$

(ii) <u>$k=2$</u>: $N^{\frac{-1-2(1-s)}{2}+\gamma} H^{1/2} H$

$\leq N^{-\frac{\alpha}{2}} F^{1/2} F \leq F$ $\quad$ choose $\gamma \leq \frac{1-s}{2}$

<u>$k=1$</u>: $N^{\frac{-1-(1-s)}{2}+\gamma} H$

$\leq H \leq F$.

(iii) $N^\theta (1 + H^{\frac{3}{4}})$

$$\lesssim N^\theta F^{\frac{3}{4}} = \underbrace{N^\theta F^{-\frac{1}{4}}}_{\lesssim 1} \cdot F$$

if $\theta \le \frac{\beta}{4}$ i.e. given $\beta > 0$, choose $\theta \ll 1$.

$$\Rightarrow H(t) \le H(t_0) + C_{M,T,s}\left(1 + \int_0^t F^{1+c(\log N)^{-1}}\right) A \log N$$

- Let $\beta < \tilde{\beta} < \alpha$

  consider $F(t) = N^{\tilde{\beta} + A(t - t_0)}$

  - $F(t_0) = N^{\tilde{\beta}} > N^\beta \ge H(t_0)$

- Suppose that $\exists\ t < T$ s.t. $t - t_0 < \frac{\alpha - \tilde{\beta}}{A}$

  and $F(t) \le H(t)$

(Otherwise, we would have
$H(t) \le F(t) \le N^{\tilde{\beta} + \alpha - \tilde{\beta}} = N^\alpha$, $\forall\ t - t_0 < \frac{\alpha - \tilde{\beta}}{A}$

- By continuity of H and F,
  $\exists\ t_*$ with $t_* - t_0 < \frac{\alpha - \tilde{\beta}}{A}$
  s.t. $H(t_*) = F(t_*)$

$$\Rightarrow \quad N^{\tilde\beta + \lambda(t_* - t_0)}$$

$$\le N^{\beta} + c\Big(1 + N^{\tilde\beta(1+c(\log N)^{-1})} \times \int_{t_0}^{t_*} N^{\lambda(1+c(\log N)^{-1})(t-t_0)}\,dt\Big)$$

$$\times A \log N$$

$$\le N^{\tilde\beta + \lambda(t_* - t_0)}$$

$$\times \Big\{ \frac{1}{N^{\tilde\beta - \beta} N^{\lambda(t_* - t_0)}} + c\,\frac{A\log N}{N^{\tilde\beta + \lambda(t_* - t_0)}}$$

$$+ c A \cancel{(\log N)}\, N^{\tilde\beta c (\log N)^{-1}} \cdot \cancel{\frac{1}{N^{\lambda(t_*-t_0)}}}$$

$$\times \underbrace{\int_{t_0}^{t_*} N^{\lambda(1+c(\log N)^{-1})(t-t_0)}\,dt}$$

$$= \frac{N^{\lambda(\cancel{1}+c(\log N)^{-1})(t_*-t_0)} - 1}{\lambda(1+c(\log N)^{-1})\cancel{\log N}} \quad \text{drop}$$

$$\Rightarrow 1 \leq \underbrace{\boxed{\frac{1}{N^{\tilde\beta-\beta} N^{\lambda(t_* - t_0)}} + \frac{c\,A \log N}{N^{\tilde\beta} N^{\lambda(t_* - t_0)}}}}_{\leq 1/2 \quad \text{if } A \lesssim N^{\beta}.}$$

$$+ \underbrace{c\,A \frac{N^{\tilde\beta c(\log N)^{-1}} N^{\lambda c (\log N)^{-1}(t_* - t_0)}}{\lambda(1 + c(\log N)^{-1})}}_{= cA \frac{e^{c\tilde\beta + c\lambda(t_* - t_0)}}{\lambda(1 + c(\log N)^{-1})}} \qquad \left(N^{c(\log N)^{-1}} = e^{c}\right.$$

$$t_* - t_0 \leq \frac{\alpha - \tilde\beta}{\lambda}$$

$$\Rightarrow \frac{1}{2} \leq \frac{c\,A\,e^{c\alpha}}{\lambda(1 + c(\log N)^{-1})}$$

$$\leq c' \frac{A}{\lambda} e^{c\alpha} \Rightarrow \otimes \quad \text{by choosing } \frac{A}{\lambda} = \kappa \ll 1.$$

$$\Rightarrow H(t) \leq F(t) \leq N^{\alpha}, \; \forall \; t - t_0 \leq \frac{\alpha - \tilde\beta}{\lambda}$$

$$= \frac{\tau}{A}.$$

$\square$

Prop 13: (almost a.s. GWP): Let $s > 4/5$.

Given $T, \varepsilon > 0$ (unrelated!!), $\exists\, \Omega_{T,\varepsilon} \subset \Omega$

s.t. ① $P(\Omega^c_{T,\varepsilon}) < \varepsilon$

② $\forall\, \omega \in \Omega_{T,\varepsilon}$,

$\exists !$ soln u to the defocusing cubic r-SNLW on $[0,T]$ of the form $u = v + \Psi$

satisfying $\|v\|_{L^\infty_T H^s_x} \le C(s, T, \varepsilon)$.

Rmk: a.a.s. GWP $\underset{\text{Borel-Cantelli}}{\Rightarrow}$ a.s. GWP (i.e. GWP as an SPDE).

Pf: Fix $N_0 = N_0(s)$

$$N_{k+1} = N_k^{\sigma}, \quad \sigma > 1$$

s.t.

$$(+) \quad \boxed{N_{k+1}^{2(1-s)} N_k^{\alpha} + N_k^{2\alpha} \ll N_{k+1}^{\beta}}$$

$\Rightarrow$ Need $\beta > 2(1-s)$

$2(1-s) < \beta < \alpha \le 1 - 3(1-s)$

$\Rightarrow$ $s > 4/5$

· Suppose $H(I_{N_k} v)(t) \le N_k^{\alpha}$

Then,

$$H(I_{N_{k+1}} v)(t) \lesssim \| I_{N_{k+1}} \vec{v} \|_{\mathcal{H}^1}^2 + \| I_{N_{k+1}} v \|_{L^4}^4$$

$\vec{v} = (v, \partial_t v)$

$$\lesssim N_{k+1}^{2(1-s)} \| \vec{v} \|_{\mathcal{H}^s}^2 + \underbrace{\| I_{N_{k+1}} v \|_{H^{1/2}}^4}$$

$(\|f\|_{H^s} \lesssim \|If\|_{H^1}$

$$\le \| v \|_{H^{1/2}}^4 \lesssim \| I_{N_k} v \|_{H^1}^4$$

$$\lesssim N_{k+1}^{2(1-s)} H(I_{N_k} v) + H^2(I_{N_k} v)$$

$$\lesssim N_{k+1}^{2(1-s)} N_k^{\alpha} + N_k^{2\alpha} \overset{+}{\ll} N_{k+1}^{\beta}$$

++ $\Rightarrow$ $H(I_{N_{k+1}} v)(t) \le N_{k+1}^{\beta}$

Let $\Omega_\Lambda(N) = \{ A(N) \le \Lambda \}$

$\Rightarrow$ By Lemma 10,

$$P(\Omega_\Lambda(N)^c) \le \frac{C^{\log N} T^{\frac{\log N}{2}+1}}{\Lambda^{\log N}} \lesssim N^{\overbrace{\log C_{T,s} - \log \Lambda}^{<0}}$$

<0 by choosing $\Lambda \gg 1$

Define $\Omega_\Lambda = \bigcap_{k \in \mathbb{Z}_{\ge 0}} \Omega_\Lambda(N_k)$

$$\Rightarrow P(\Omega_\Lambda^c) \le \sum_k N_k^{\log C_{T,s} - \log \Lambda} \lesssim N_0^{\log C_{T,s} - \log \Lambda}$$

$\to 0$ as $\Lambda \to \infty$.

Choose $M(T,\varepsilon)$, $\Lambda(T,\varepsilon) \gg 1$ s.t.

$$P(\Omega_\Lambda^c) + P(\Omega_{M,\gamma,\theta}^c) < \varepsilon$$

Choose $k_0 \gg 1$ s.t.

- $\Lambda < N_{k_0}^{\delta}$
- $H(I_{N_{k_0}} v)(0) \le N_{k_0}^{\beta}$.

$$\overset{\text{Prop 12}}{\Longrightarrow} \quad H(I_{N_{k_0}} v)(t) \le N_{k_0}^{\alpha}, \quad \forall\, t < \frac{\tau}{A(N_{k_0})}$$

With $t = \frac{\tau}{\Lambda}$. (Recall $A(N_k) \le \Lambda$ on Ω_Λ)

$$\overset{(++)}{\Longrightarrow} \quad H(I_{N_{k_0+1}} v)\left(\frac{\tau}{\Lambda}\right) \le N_{k_0+1}^{\beta}.$$

$$\overset{\text{Prop 12}}{\Longrightarrow} \quad H(I_{N_{k_0+1}} v)\left(2\frac{\tau}{\Lambda}\right) \le N_{k_0+1}^{\alpha}$$

$$\overset{(++)}{\Longrightarrow} \quad H(I_{N_{k_0+2}} v)\left(2\frac{\tau}{\Lambda}\right) \le N_{k_0+2}^{\beta}$$

$$\overset{\text{Prop 12}}{\Longrightarrow} \quad \cdots$$

Iterate the argument $\left[\frac{\Lambda T}{\tau}\right] + 1$ times.

(Note: increment is indep of N.)

$\Rightarrow$ Given $t \in \left[(j-1)\frac{\tau}{\Lambda}, j\frac{\tau}{\Lambda}\right]$, we have

$$\|\vec{v}(t)\|_{\mathcal{H}^s} \lesssim H(I_{N_{k_0+j-1}} v)^{1/2}(t)$$

$$\leq N_{k_0+j-1}^{\alpha/2} \quad \left(\leq N_{k_0+[\frac{\Lambda T}{\tau}]-1}^{\alpha/2} \right)$$

$k_0, \alpha, \Lambda, \tau$ depend only on s, T, and ε.

$\boxtimes$

Rmk: A standard application of the I-method:

Given $T \gg 1$, choose $N = N(T) \gg 1$.

but in our argument, we needed to choose

an increasing seq N_{k_0+j}, $j = 0, 1, \dots, [\frac{\Lambda T}{\tau}]$

$\Leftarrow$ much more subtle.

J. Forlano on BBM '18.

- GWP of the defocusing cubic SNLW on $\mathbb{R}^2$
  L. Tolomeo '18.

Chap 2: quadratic SNLW in 3-d

$$\begin{cases} (\partial_t^2 + 1 - \Delta) u + u^2 = \xi \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases} \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^3.$$

$d = 3 \Rightarrow \xi$ has spatial regularity $-\frac{3}{2} - \varepsilon$.

• Parabolic case:

(SNLH) $(\partial_t + 1 - \Delta) u + u^2 = \xi$

Da Prato - Debussche: $u = v + \mathbf{!}$

$$\mathbf{!} = \mathcal{I}(\xi)$$

vertex $\bullet = \xi$

edge $=$ Duhamel integral operator $\mathcal{I}$.
$= (\partial_t + 1 - \Delta)^{-1}$

· The heat Duhamel integral operator $\mathcal{I}$ gains 2 derivatives (with dt') but with $dW(t')$, only 1 derivative
($\Leftarrow$ 2 derivatives in the second moment.)

$\Rightarrow \mathbf{!} \sim -\frac{1}{2}-$ We only keep track of spatial regularities.

Rules

- A product of functions of reg. s_1 and s_2 is defined if $s_1 + s_2 > 0$. When $s_1 > 0$ and $s_1 \geq s_2$, the resulting prod has regularity s_2.

- A product of stochastic objects (not depending on the unknown) is always well defined, possibly with a renormalization. The product of stoch. obj. of reg s_1 and s_2 has regularity

$$\min(s_1, s_2, s_1 + s_2)$$

$$(\text{SNLH}) \overset{D\text{-}D}{\Longrightarrow} (\partial_t + 1 - \Delta) v = -(v + \mathsf{f})^2 \qquad \text{renorm } \mathsf{f}^2 \rightsquigarrow \mathsf{V}$$

$$\text{“}=\text{”} - v^2 - 2v\mathsf{f} - \mathsf{V}$$

$$\uparrow$$
$$-1-$$

$\Rightarrow$ expect $v \sim 1- = (-1-) + 2$.

$$\underset{1-}{v} \quad \underset{-\frac{1}{2}-}{\mathsf{f}} \qquad (1-) + (-\tfrac{1}{2}-) > 0$$

$\Rightarrow$ The product $v\mathsf{f}$ and hence all the terms on RHS are well defined.

$\Rightarrow$ run a contraction argument in $C_T C_x^{1-}$

Back to the wave case:

$$ϙ = \mathcal{I}(\xi) \sim -\tfrac{1}{2}-$$

Here, $\mathcal{I} = (\partial_t^2 + 1 - \Delta)^{-1}$

(corresp to the forward fund. soln.)

Wick power: $\curlyvee_N = ϙ_N^2 - \sigma_N$ with $ϙ_N = P_{\le N} ϙ$

$\cdot$ $\sigma_N = \mathbb{E}[ϙ^2(t,x)] \sim tN.$

$$\Rightarrow \quad \curlyvee_N \to \curlyvee \text{ in } C_T W_x^{-1,\infty} \text{ a.s.}$$

2nd order stoch. obj:

$$\Upsilon = \mathcal{I}(\curlyvee) = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} \curlyvee(t')\,dt'$$

A naive "parabolic thinking" yields

$$0- = 2(-\tfrac{1}{2}-) + 1.$$

$\Leftarrow$ BAD. We need to use the explicit product structure and exhibit multilinear dispersive smoothing

Prop 14: $\mathbf{Y}_N = \mathcal{I}(\mathbf{V}_N) \to \mathbf{Y}$

in $C_T W_x^{\boxed{\frac{1}{2}-\varepsilon},\infty} \cap C_T^1 W_x^{-1-\varepsilon,\infty}$, a.s.

i.e. $\mathbf{Y} \sim \frac{1}{2}-$ $\Leftarrow$ extra $\frac{1}{2}-$ smoothing

---

Seconder order expansion:

$$u = \mathbf{1} - \mathbf{Y} + v.$$

"at least as smooth as $\mathbf{Y}$"

$$(\text{SNLW}) \Rightarrow (\partial_t^2 + 1 - \Delta)v = -(v + \mathbf{1} - \mathbf{Y})^2 + \mathbf{V}$$

$$= -(v - \mathbf{Y})^2 - 2v\mathbf{1} + \underbrace{2\mathbf{1}\mathbf{Y}}_{-\frac{1}{2}-}$$

$\Rightarrow$ expect $v \sim \frac{1}{2}-$

$\Rightarrow$ $v\mathbf{1}$: $(\frac{1}{2}-) + (-\frac{1}{2}-) < 0$

The product does not make sense.

So, the second order expansion is NOT enough.

- Littlewood-Paley projection: P_j onto $\{|n| \sim 2^j\}$.

$$f = \sum_{j=0}^{\infty} P_j f.$$

- Paraproduct decomposition (Bony '81).

$$fg = f \textcircled{<} g + f \textcircled{=} g + f \textcircled{>} g$$

$$= \sum_{j<k-2} P_j f P_k g + \sum_{|j-k|\le 2} P_j f P_k g + \sum_{k<j-2} P_j f P_k g.$$

- $f \textcircled{<} g$ = paraproduct of g by f (g = high freq)

  always well defined as a distribution of regularity $\min(s_2, s_1+s_2)$

- $f \textcircled{=} g$ = resonant product

  In general, well defined only if $s_1+s_2>0$.

  $\Leftarrow$ The resonant prod is the source of difficulty in making sense of a nonlinearity.

- Set $\textcircled{\geq} = \textcircled{>} + \textcircled{=}$.

Further decomposition:

Paracontrolled ansatz: $v = \underset{s_1,}{X} + \underset{s_2,}{Y} \qquad 0 < s_1 < s_2$

(SMLW2)
$$
\begin{aligned}
(\partial_t^2 + 1 - \Delta) X &= -2\,(X + Y - \curlyvee) \varolessthan \Psi \\
(\partial_t^2 + 1 - \Delta) Y &= -(X + Y - \curlyvee)^2 - 2(X + Y - \curlyvee) \varogreaterthan \Psi
\end{aligned}
$$

· Why called a paracontrolled ansatz?

A distribution f is said to be paracontrolled (by a given reference distribution g) if $\exists\ f'$ s.t.

$$f = f' \varolessthan g + \underset{\uparrow \text{ smoother}}{h}$$

(SNLW2) says $\square v = \square X + \square Y$ is paracontrolled.

Expect · $X \sim \frac{1}{2}- = (-\frac{1}{2}-) + 1.$

For now, ignore $\circeq$ in the Y-eqn.

$$(X + Y - \curlyvee) \varogreaterthan \Psi \sim 0-$$

$\Rightarrow Y \sim 1-$

$\Rightarrow Y \circeq \Psi$ makes sense as long as $s_2 > 1/2$.

$$\Upsilon^{\bullet}_{\textcircled{=}} = \Upsilon \mathbin{\textcircled{=}} \mathrm{I}$$

$0- = (\tfrac{1}{2}-) + (-\tfrac{1}{2}-)$ without renormalization

We did not expect any further renorm. since it was not needed for SNLH.

In order to prove this, we need to exploit dispersion at a multilinear level.
(Otherwise, it looks as if there is a log divergence.

Prop 15: $\Upsilon^{\bullet}_{\textcircled{=}\,N} = \Upsilon_N \mathbin{\textcircled{=}} \mathrm{I}_N \to \Upsilon^{\bullet}_{\textcircled{=}}$ in $C_T W_x^{-\varepsilon,\infty}$ a.s.

- $X \mathbin{\textcircled{=}} \mathrm{I}$?

$(\tfrac{1}{2}-) + (-\tfrac{1}{2}-) < 0 \Rightarrow$ does not make sense as it is.

Idea: Use the structure of X_1, i.e.

$$X(t) = \underbrace{\partial_t S(t) X_0 + S(t) X_1}_{= \vec{S}(t)(X_0, X_1)} - 2\mathcal{I}\big((X + \Upsilon - \Upsilon) \mathbin{\textcircled{<}} \mathrm{I}\big)$$

Lem 16: $s_1 > 0$. $(X_0, X_1) \in \mathcal{H}^{s_1}(\mathbb{T}^3)$.

$$Z_N = \big(\vec{S}(t)(X_0, X_1)\big) \textcircled{=} \dot{\imath}_N$$

$$\downarrow$$

$$Z = \big(\vec{S}(t)(X_0, X_1)\big) \textcircled{=} \dot{\imath}$$

← Need to iclude Z in an enhanced data set.

in $C_T H_x^{s_1 - \frac{1}{2} - \varepsilon}$, a.s.

(A straightforward computation with the Wiener chaos estimate and a direct application of Kolmogorov's continuity criterion.

Note: Set of prob 1 depends on the initial data (X_0, X_1).

In the parabolic setting, at this point, one would introduce (smoother) commutators:

BAD

$$\big[\mathcal{I}\big((X + Y - Y^{\dot{\imath}}) \textcircled{<} \dot{\imath}\big)\big] \textcircled{=} \dot{\imath}$$

$$= \big[(X + Y - Y^{\dot{\imath}}) \textcircled{<} \mathcal{I}(\dot{\imath})\big] \textcircled{=} \dot{\imath} + \boxed{\mathrm{com}_1} \textcircled{=} \dot{\imath}$$

$$= (X + Y - Y^{\dot{\imath}}) \textcircled{<} \big\{\underbrace{\mathcal{I}(\dot{\imath}) \textcircled{=} \dot{\imath}}_{\substack{\text{explicitly known} \\ \text{stoch obj. of reg } 0-}}\big\} + \boxed{\mathrm{com}_1} \textcircled{=} \dot{\imath} + \mathrm{com}_2$$

Bad News: com_1 is NOT smooth in the dispersive setting!!

Main idea: directly study the following paracontrolled operator (and its res product with Ψ).

Given $w \in C(\mathbb{R}_+; H^{s_1}(\mathbb{T}^3))$ with $0 < s_1 < \frac{1}{2}$,

define $\boxed{\mathcal{J}_{\circledless}(w)(t) = \mathcal{I}(w \circledless \Psi)(t)}$

$$= \sum_{j < k-2} \mathcal{I}(P_j w \cdot P_k \Psi)$$

$$= \sum_{n \in \mathbb{Z}^3} e_n \sum_{\substack{n = n_1 + n_2 \\ |n_1| \ll |n_2|}} \int_0^t \frac{\sin((t-t')\langle n \rangle)}{\langle n \rangle} \widehat{w}(t', n_1) \widehat{\Psi}(t', n_2)\, dt'$$

signifies the paraproduct $\circledless$.

Goal: Make sense of $\mathcal{J}_{\circledless}(w) \circledeq \Psi$

with $w = X + Y - \Psi$.

- Divide $\mathcal{J}_{\circledless}$ into good and bad parts

Fix $\theta > 0$ small

$\mathcal{J}^{(1)}_{\circledless}$ = restriction of $\mathcal{J}_{\circledless}$ onto $\{|n_1| \gtrsim |n_2|^\theta\}$

$\mathcal{J}^{(2)}_{\circledless} = \mathcal{J}_{\circledless} - \mathcal{J}^{(1)}_{\circledless}$.

Namely,

$$\mathcal{J}^{(1)}_{\circledcirc}(w)(t) = \sum_{n\in\mathbb{Z}^3} e_n \sum_{\substack{n=n_1+n_2 \\ |n_2|^\theta \lesssim |n_1| \ll |n_2|}} \int_0^t \frac{\sin((t-t')\langle n\rangle)}{\langle n\rangle} \widehat{w}(t',n_1)\widehat{1}(t',n_2)\,dt'.$$

point: $|n_1|$ is NOT too small. $\rightarrow$ $|n| \sim |n_2|$

$$\langle n\rangle^{\frac{1}{2}+2\varepsilon} \frac{1}{\langle n\rangle} \lesssim \langle n_1\rangle^{\frac{4\varepsilon}{\theta}} \langle n_2\rangle^{-\frac{1}{2}-2\varepsilon}$$

$$\lesssim \langle n_1\rangle^{s_1-\varepsilon} \langle n_2\rangle^{-\frac{1}{2}-2\varepsilon}$$

by choosing $\varepsilon = \varepsilon(s_1,\theta) > 0$ suff. small.

Lem 17: $0 < s_1 < \frac{1}{2}$. Given small $\theta > 0$,
$\exists$ small $\varepsilon = \varepsilon(s_1,\theta) > 0$ s.t. given any
$\boxminus \in C(\mathbb{R}_+ ; W^{-\frac{1}{2}-\varepsilon,\infty}(\mathbb{T}^3))$,

the paracontrolled operator

$$\mathcal{J}^{(1),\boxminus}_{\circledcirc} = \mathcal{J}^{(1)}_{\circledcirc} \text{ with } 1 \text{ replaced by } \boxminus$$

belongs to $\mathcal{L}_2 = \mathcal{L}\left(C_T H^{s_1}_x ; C_T H^{\frac{1}{2}+2\varepsilon}_x\right)$

$\uparrow$

· following the notation from GKO'18.

- When $|n_1| \ll |n_2|^\theta$, the positive reg of w does not help.

We directly study $\mathcal{J}_{\textcircled{<},\textcircled{=}}$.

$$\boxed{\mathcal{J}_{\textcircled{<},\textcircled{=}}(w)(t) = \mathcal{J}^{(2)}_{\textcircled{<}}(w) \textcircled{=} \varphi(t)}$$

$$= \sum_{n \in \mathbb{Z}^3} e_n \int_0^t \sum_{n_1 \in \mathbb{Z}^3} \widehat{w}(t', n_1) A_{n,n_1}(t,t')\,dt'.$$

$$A_{n,n_1}(t,t') = \mathbb{1}_{[0,t]}(t') \sum_{\substack{n - n_1 = n_2 + n_3 \\ |n_1| \ll |n_2|^\theta \\ |n_1+n_2| \sim |n_3|}} \frac{\sin((t-t')\langle n_1+n_2\rangle)}{\langle n_1+n_2\rangle} \widehat{\varphi}(t', n_2)\widehat{\varphi}(t,n_3)$$

— resonant product

$$= \mathbb{1}_{[0,t]}(t') \sum \frac{\sin((t-t')\langle n_1+n_2\rangle)}{\langle n_1+n_2\rangle}\Big(\widehat{\varphi}(t', n_2)\widehat{\varphi}(t,n_3) - \mathbb{1}_{n_2+n_3=0}\,\sigma_{n_2}(t,t')\Big)$$

$$+ \mathbb{1}_{[0,t]}(t')\cdot \mathbb{1}_{n=n_1} \sum_{\substack{n_2 \in \mathbb{Z}^3 \\ |n| \ll |n_2|^\theta}} \frac{\sin((t-t')\langle n+n_2\rangle)}{\langle n+n_2\rangle} \sigma_2(t,t')$$

$$=: A^{(1)}_{n,n_1}(t,t') + \underbrace{A^{(2)}_{n,n_2}(t,t')}_{\text{deterministic counter term}}$$

More difficult!!

Here, $0 \le t_2 \le t_1$,

$$\sigma_n(t_1,t_2) = \mathbb{E}\,[\widehat{\varphi}(t_1,n)\widehat{\varphi}(t_2,n)]$$

$$= \frac{\cos((t_1-t_2)\langle n\rangle)}{2\langle n\rangle^2}\, t_2 + O\Big(\frac{1}{\langle n\rangle^3}\Big).$$

By exploiting dispersion (stationary phase) and symmetrization ($\underbrace{n_2 \leftrightarrow -n_2}$)

i.e. the order of summation matters
$\Rightarrow$ only conditionally convergent.

Prop 18: $s_2 < 1$. $\exists$ small $\theta(s_2) > 0$ and $\varepsilon > 0$ s.t.

$\mathcal{J}_{\circledless, \circleddash}$ belongs to the class

$$\mathcal{L}_1 = \mathcal{L}\left(C_T L^2_x \cap C^1_T H^{-1-\varepsilon}_x \,;\, C_T H^{s_2-1}_x\right).$$

Moreover, $\mathcal{J}^N_{\circledless, \circleddash}$ (replacing $\dagger$ by $\dagger_N$) converges a.s. to $\mathcal{J}_{\circledless, \circleddash}$ in $\mathcal{L}_1$.

Final form:

$$(\partial_t^2 + 1 - \Delta) X = -2(X + Y - \Upsilon) \circledless \dagger$$

(SNLW3)

$$\begin{aligned}(\partial_t^2 + 1 - \Delta) Y = &-(X + Y - \Upsilon)^2 - 2(X + Y - \Upsilon) \circledgtr \dagger \\ &- 2Y \circleddash \dagger + 2 \,\Upsilon_{\ominus} - 2\mathcal{Z} \\ &+ 4 \mathcal{J}^{(1)}_{\circledless} (X + Y - \Upsilon) \circleddash \dagger \\ &+ 4 \mathcal{J}_{\circledless, \circleddash} (X + Y - \Upsilon)\end{aligned}$$

$$(X, \partial_t X, Y, \partial_t Y)\big|_{t=0} = (X_0, X_1, Y_0, Y_1).$$

Thm 19: $\frac{1}{4} < s_1 < \frac{1}{2} < s_2 \le s_1 + \frac{1}{4}$.

$\exists\ \theta = \theta(s_2) > 0$ and $\varepsilon = \varepsilon(s_1, s_2, \theta) > 0$ s.t.

if (i) $\mathbf{1}, \curlyvee, \curlyvee\!\!\!\circ$ in $C(\mathbb{R}_+; W_x^{s,\infty})$

$s = -\frac{1}{2}-, \frac{1}{2}-, 0-$.

and

$\curlyvee \in C^1(\mathbb{R}_+; W_x^{-1-\varepsilon})$

(ii) $\mathcal{Z} \in C(\mathbb{R}_+; H_x^{s_1 - \frac{1}{2} - \varepsilon})$.

(iii) $\mathcal{J}_{\circledcirc, \ominus} \in \mathcal{L}_1$,

NOT random

then the system (SNLW3) is locally well-posed.

More precisely, given $(X_0, X_1, Y_0, Y_1) \in \mathcal{H}^{s_1} \times \mathcal{H}^{s_2}$,

$\exists\ T$ and unique soln (X, Y) in the class

$$Z_T^{s_1, s_2} = X_T^{s_1} \times Y_T^{s_2}$$

$$\subset C([0,T]; H^{s_1} \times H^{s_2}) \cap C^1([0,T]; H^{s_1 - 1} \times H^{s_2 - 2})$$

depending continuously on the enhanced data set

$$\Xi = (X_0, X_1, Y_0, Y_1, \mathbf{1}, \curlyvee, \curlyvee\!\!\!\circ, \mathcal{Z}, \mathcal{J}_{\circledcirc\ominus})$$

in the class

$$\mathcal{X}_T^{s_1, s_2, \varepsilon} = \mathcal{H}^{s_1} \times \mathcal{H}^{s_2} \times C_T W_x^{-\frac{1}{2} - \varepsilon, \infty} \times \left(C_T W_x^{\frac{1}{2} - \varepsilon} \cap C_T^1 W_x^{-1-\varepsilon, \infty}\right)$$
$$\times\ C_T W_x^{-\varepsilon, \infty} \times C_T H^{s_1 - \frac{1}{2} - \varepsilon} \times \mathcal{L}_1$$

- $$u_N = \overset{\circ}{\mathsf{Y}}_N - \overset{\vee}{\mathsf{Y}}_N + \underbrace{X_N + Y_N}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$u = \overset{\circ}{\mathsf{Y}} - \overset{\vee}{\mathsf{Y}} + X + Y \quad \text{in } C_T H_x^{-\frac{1}{2}-\varepsilon}$$

Pf of Thm 19 : Duhamel formulation.

$$X(t) = \Phi_1(X,Y)(t)$$

$$= \vec{S}(t)(X_0, X_1) - 2\int_0^t S(t-t')[(X+Y-\overset{\vee}{\mathsf{Y}}) \circledcirc \overset{\circ}{\mathsf{Y}}](t')\,dt'.$$

$$Y(t) = \Phi_2(X,Y)(t)$$

$$= \vec{S}(t)(Y_0, Y_1) - \int_0^t S(t-t')[\cdots](t')\,dt'.$$

Strichartz estimates : $0 \le s \le 1$

We say a pair (q,r) is s-admissible

$(\tilde{q}, \tilde{r})$ is dual s-admis.

(i.e. $(\tilde{q}', \tilde{r}')$ is $(1-s)$-admis.)

if $1 \le \tilde{q} < 2 < q \le \infty$, $1 < \tilde{r} \le 2 \le r < \infty$,

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s = \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} - 2 \qquad \leftarrow \text{scaling condition}$$

$$\left.\begin{aligned} \frac{1}{q} + \frac{d-1}{2r} &\le \frac{d-1}{4} \\ \frac{2}{\tilde{q}} + \frac{d-1}{2\tilde{r}} &\ge \frac{d+3}{4} \end{aligned}\right\} \quad \text{admissibility cond.}$$

Then, a soln u to

$$\begin{cases} (\partial_t^2 + 1 - \Delta) u = f \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

satisfies

$$\|(u, \partial_t u)\|_{C_T \mathcal{H}_x^s} + \|u\|_{L_T^q L_x^r}$$

$$\lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|f\|_{L_T^{\tilde{q}'} L_x^{\tilde{r}'} + L_x^1 H_x^{s-1}}$$

for $0 \le T \le 1$.

(q, r), s-admis
$(\tilde{q}, \tilde{r})$, dual s-admis.

($\Leftarrow$ follows from the Strichartz estimates on $\mathbb{R}^d$ and finite speed of propagation.

· Back to the proof of Thm 19: Define the Strichartz space

$$X_T^{s_1} = C_T H_x^{s_1} \cap C_T^1 H_x^{s_1 - 1} \cap L_T^8 W_x^{s_1 - \frac{1}{4}, \frac{8}{3}} \qquad (8, \tfrac{8}{3}),\ \tfrac{1}{4}\text{-admis}$$

$$Y_T^{s_2} = C_T H_x^{s_2} \cap C_T^1 H_x^{s_2 - 1} \cap L_T^4 W_x^{s_2 - \frac{1}{2}, 4} \qquad (4, 4),\ \tfrac{1}{2}\text{-admis}$$

$$Z_T^{s_1, s_2} = X_T^{s_1} \times Y_T^{s_2}$$

$$\bullet\ \|\Phi_1(X,Y)\|_{X_T^{s_1}} \lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + \underbrace{\|(X+Y-\Psi)\mathbin{\text{\textcircled{<}}}\rho\|_{L^1_T H^{s_1-1}_x}}$$

$$\lesssim T\,\|X+Y-\Psi\|_{L^\infty_T L^2_x}\,\|\rho\|_{L^\infty_T W^{-\frac{1}{2}-\varepsilon,\infty}_x}$$

$$\lesssim \|(X_0,X_1)\|_{\mathcal{H}^{s_1}} + T\big(1+\|(X,Y)\|_{Z_T^{s_1,s_2}}\big)$$

if $s_1-1<-\frac{1}{2}-\varepsilon$.

<u>Y-equation</u>:

$$-2(X+Y-\Psi)\mathbin{\text{\textcircled{>}}}\rho - 2Y\mathbin{\text{\textcircled{=}}}\rho + 2\,\Psi_\circ + \mathbb{Z}$$

$$+\ \mathcal{J}^{(1)}_{\text{\textcircled{<}}}(X+Y-\Psi)\mathbin{\text{\textcircled{=}}}\rho$$

$$\Rightarrow \|\cdots\|_{Y_T^{s_1}} \lesssim \|(Y_0,Y_1)\|_{\mathcal{H}^{s_2}} + T\big(1+\|(X,Y)\|_{Z_T^{s_1,s_2}}\big).$$

if $\quad \frac{1}{2}<s_2<\min(1,\ s_1+\frac{1}{2})$

· Also

$$\Big\|\int_0^t S(t-t')\,\mathcal{J}_{\text{\textcircled{<}},\text{\textcircled{=}}}(X+Y-\Psi)(t')\,dt'\Big\|_{Y_T^{s_2}}$$

$$\lesssim \|\mathcal{J}_{\text{\textcircled{<}},\text{\textcircled{=}}}(X+Y-\Psi)\|_{L^1_T H^{s_2-1}_x}$$

$$\lesssim T\,\|X+Y-\Psi\|_{L^\infty_T L^2_x\,\cap\, C^1_T H^{-1-\varepsilon}_x}$$

$$\lesssim T\big(1+\|(X,Y)\|_{Z_T^{s_1,s_2}}\big).$$

$$\left\| \int_0^t S(t-t')(X+Y-\overset{\bullet}{Y})^2(t')\,dt' \right\|_{Y_T^{s_2}}$$

$$\overset{\text{Strichartz}}{\lesssim} \| \langle \nabla \rangle^{s_2 - \frac{1}{2}} (X + Y - \overset{\bullet}{Y})^2 \|_{L^{4/3}_{T,x}}$$

$$\overset{\text{frac. Leib.}}{\lesssim} T^{\frac{1}{4}} \Big(\| \langle \nabla \rangle^{s_2 - \frac{1}{2}} X \|^2_{L^8_T L^{8/3}_x} + \| \langle \nabla \rangle^{s_2 - \frac{1}{2}} Y \|^2_{L^4_{T,x}} + \| \langle \nabla \rangle^{s_2 - \frac{1}{2}} \overset{\bullet}{Y} \|^2_{L^\infty_{T,x}} \Big)$$

Need $\le s_1 - \frac{1}{4}$

$$\lesssim T^{\frac{1}{4}} \left(1 + \| (X,Y) \|^2_{Z_T^{s_1,s_2}} \right).$$

if $s_2 \le \min(1-\varepsilon,\ s_1 + \frac{1}{4})$.

$$\Rightarrow \| \Phi(X,Y) \|_{Z_T^{s_1,s_2}} \lesssim \| (X_0, X_1, Y_0, Y_1) \|_{\mathcal{H}^{s_1} \times \mathcal{H}^{s_2}} + T^\theta \left(1 + \| (X,Y) \|^2_{Z_T^{s_1,s_2}} \right)$$

for some $\theta > 0$.

Similarly,

$$\| \Phi(X,Y) - \Phi(\tilde{X}, \tilde{Y}) \|_{Z_T^{s_1,s_2}}$$

$$\lesssim T^\theta \left(1 + \| (X,Y) \| + \| (\tilde{X}, \tilde{Y}) \| \right) \| (X,Y) - (\tilde{X}, \tilde{Y}) \|_{Z_T^{s_1,s_2}}$$

$\Rightarrow$ By taking $T > 0$ suff. small,

Φ is a contraction on a ball $B_R \subset Z_T^{s_1,s_2}$

$\boxtimes$

Key point of the proof of Prop 14 on $\overset{\Psi}{Y}$

In using Lemma 2, we bound

$$\mathbb{E}\big[|\widehat{\overset{\Psi}{Y}}(t,n)|^2\big]$$

$$= 4 \sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \int_0^t \frac{\sin(t-t_1)\langle n\rangle}{\langle n\rangle} \int_0^{t_1} \frac{\sin(t-t_2)\langle n\rangle}{\langle n\rangle} \Gamma_{n_1}(t_1,t_2) \times \Gamma_{n_2}(t_1,t_2)\, dt_2\, dt_1$$

$$+ \cdots_{n_1=n_2}$$

$$\Gamma_n(t_1,t_2) = \frac{\cos(t_1-t_2)\langle n\rangle}{2\langle n\rangle^2} t_2 + O\Big(\frac{1}{\langle n\rangle^3}\Big)$$

$\Rightarrow$ expand sines and cosines in complex exponentials

$$\sum_{\substack{n=n_1+n_2 \\ n_1 \neq \pm n_2}} \frac{e^{i(\varepsilon_1+\varepsilon_2)t\langle n\rangle}}{\langle n\rangle^2\langle n_1\rangle^2\langle n_2\rangle^2} \qquad \varepsilon_j \in \{\pm 1\}$$

$$\times \underbrace{\int_0^t e^{-it_1K_1(\bar{n})} \int_0^{t_1} t_2^2\, e^{-it_2K_2(\bar{n})}\, dt_2\, dt_1}$$

$$K_1(\bar{n}) = \varepsilon_1\langle n\rangle - \varepsilon_3\langle n_1\rangle - \varepsilon_4\langle n_2\rangle$$

$$K_2(\bar{n}) = \varepsilon_2\langle n\rangle + \varepsilon_3\langle n_1\rangle + \varepsilon_4\langle n_2\rangle$$

integrate in t_1 first.

$$\Big| \int_0^t t_2^2\, e^{-it_2K_2(\bar{n})} \frac{e^{-itK_1(\bar{n})} - e^{-it_2K_1(\bar{n})}}{-iK_1(\bar{n})}\, dt_2 \Big|$$

$$\leq \frac{C(T)}{(1+K_1(\bar{n}))}$$

Need to bound

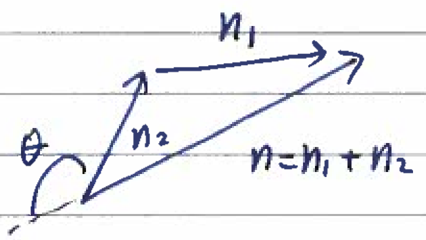
$$I = \sum_{n=n_1+n_2} \frac{1}{\langle n\rangle^2\langle n_1\rangle^2\langle n_2\rangle^2(1+|K_1(\bar{n})|)} \qquad \left(\text{WTS } \lesssim \langle n\rangle^{-4+}\right)$$

<u>BAD case</u> $(\varepsilon_1, \varepsilon_3, \varepsilon_4) = (\pm 1, \pm 1, \mp 1)$.

Assume. $|n| \gg 1$, $|n_1| \geq |n_2|$

$$|K_1(\bar{n})| = \langle n\rangle + \langle n_2\rangle - \langle n_1\rangle$$

Note: $\langle n_1\rangle \sim \langle n\rangle + \langle n_2\rangle$



Law of cosines: $|n|^2 + |n_2|^2 - |n_1|^2$

$$= 2|n||n_2|\cos(\angle(n, n_2)).$$

$$\Rightarrow |K_1(\bar{n})| = \frac{(\langle n\rangle + \langle n_2\rangle)^2 - \langle n_1\rangle^2}{\langle n\rangle + \langle n_2\rangle + \langle n_1\rangle}$$

$$= \frac{2\langle n\rangle\langle n_2\rangle + |n|^2 + |n_2|^2 - |n_1|^2 + 1}{\langle n\rangle + \langle n_2\rangle + \langle n_1\rangle}$$

$$\gtrsim \frac{|n||n_2|(1-\cos\theta)}{\langle n_1\rangle} \qquad \theta = \angle(n_2, -n)$$

Case 1: $1-\cos\theta \gtrsim 1.$ (large angle) non-resonant

$$I \lesssim \sum_{n=n_1+n_2} \frac{1}{\langle n\rangle^3 \langle n_1\rangle \langle n_2\rangle^3} \qquad \langle n_1\rangle \sim \max(\langle n\rangle, \langle n_2\rangle)$$

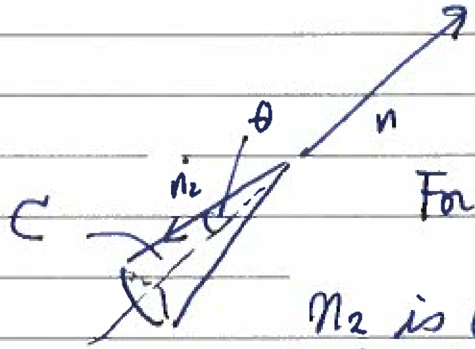
$$\lesssim \langle n\rangle^{-4+}.$$

Case 2: $1-\cos\theta \ll 1$ (close to collinear) nearly resonant

$$\Rightarrow \quad 0 \le \theta \ll 1.$$

$$\Rightarrow \quad 1-\cos\theta \sim \theta^2 \ll 1.$$

Dyadically decompose $|n_2| \sim N_2$, $N_2 \ge 1$, dyadic



For fixed $n \in \mathbb{Z}^3$,

n_2 is constrained to a cone C

$|n_2| \sim N_2$

height $\sim N_2 \cos\theta \sim N_2$

base disc of radius $\sim N_2 \sin\theta \sim N_2\theta.$

$$\Rightarrow \quad \mathrm{vol}(C) \sim N_2^3\theta^2.$$

$$\Rightarrow \quad I \le \sum_{\substack{N_2 \ge 1 \\ \text{dyadic}}} \frac{1}{\langle n\rangle^3 \max(\langle n\rangle, N_2)} \cdot N_2^3\theta^2$$

$$\lesssim \langle n\rangle^{-4+}$$

$\square$

• Prop 15 on  = 

• The proof proceeds analogously to that of Prop 14 but a bit more complicated.

New difficult term.

$\boxed{\gamma > 0 \text{ small}}$

$$\sum_{\substack{n_2 \in \mathbb{Z}^3 \\ |n+n_2| \sim |n_2| \\ |n| \ll |n_2|^\gamma}} \frac{\sin(t-t')(\langle n+n_2\rangle - \langle n_2\rangle)}{\langle n\rangle^2 \langle n+n_2\rangle \langle n_2\rangle^2} \quad \left(\text{WTS } \lesssim \langle n\rangle^{-3} \right.$$

This condition allows us to rewrite the sum as

$$\sum_{\substack{n_2 \in \mathbb{Z}^3 \\ |n| \ll |n_2|^\gamma}} \leftarrow \text{we can use symmetrization } n_2 \leftrightarrow -n_2.$$

Let $\Theta^{\pm}(n, n_2) = \langle n \pm n_2\rangle - \langle n_2\rangle \mp \dfrac{\langle n, n_2\rangle}{\langle n_2\rangle}$

$$= O\left(\frac{\langle n\rangle^2}{\langle n_2\rangle}\right)$$

$$\text{Sum} = \sum_{\substack{n_2 \in \text{``}\mathbb{Z}^3/2\text{''} \\ |n| \ll |n_2|^\gamma}} \frac{1}{\langle n\rangle^2 \langle n+n_2\rangle \langle n_2\rangle^2}$$

$$\times \left[\sin(t-t_1)\left(\frac{\langle n, n_2\rangle}{\langle n_2\rangle} + \Theta^+(n, n_2)\right) - \sin(t-t')\left(\frac{\langle n, n_2\rangle}{\langle n_2\rangle} - \Theta^-(n, n_2)\right)\right]$$

$$\overset{\text{MVT}}{\lesssim} \sum_{|n| \ll |n_2|^\gamma} \frac{1}{\langle n\rangle^2 \langle n_2\rangle^3} \left(\frac{\langle n\rangle^2}{\langle n_2\rangle}\right)^\delta \quad \text{for any } \delta \in [0,1].$$

$$\lesssim \langle n\rangle^{-3+} \qquad \square$$

On Prop 18 for $J_{<}$, $J_{=}$

The difficult part $A^{(2)}$

- symmetrization : $n_2 \leftrightarrow -n_2$
- integration by parts in time to handle
  $$\sin(t-t')(\langle n+n_2\rangle + \langle n_2\rangle).$$